PARAMETRIC DISTRIBUTION FAMILIES USEFUL FOR MODELLING NON-LIFE INSURANCE PAYMENTS DATA. TAIL BEHAVIOUR

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Abstract

The present paper describes a series of parametric distributions used for modeling non-life insurance payments data. Of those listed, special attention is paid to the transformed Beta distribution family. This distribution as well as those which are obtained from it (special cases of four-parameter transformed Beta distribution) are used in the modeling of high costs, or even extreme ones. In the literature it follows the tail behaviour of distributions depending on the parameters, because the insurance payments data are tipically highly positively skewed and distributed with large upper tails. In the paper is described the tail behavior of the distribution in the left and right side respectively, and deduced from it, a general case. There are also some graphs of probability density function for one of the transformed Beta family members, which comes to reinforce the comments made.

Keywords: non-life insurance payments data, actuary, parametric distribution, transformed Beta distribution, tail distribution behaviour.

Introduction

The insurance companies are responsible for covering claims which are aleatory in character. The quantum of claims can hardly be known, it can only be assessed through calculations based on the probability theory. Hence, the significance of a comprehensive information system comprising data and long-term assessments of the frequency and degree of risks has emerged. Based on such a system, one can approximate the evolution of potential payments to cover the value of claims and of insured accounts, as well as of the level premium system, being also able to adopt decisions linked to the initiation of new supplemental insurance programs to protect assets, people and liability insurance, or to the functioning and organizational structure of the insurance company.

An insurance company resembles a living organism, comprising several departments that must perform vital functions for the insured person to successfully survive. One of the main departments of such a company is the claims control department designed to prevent claims whenever possible and to diminish those that cannot be avoided. The claims control has always been an important function ever since the early days of the insurance system, which is gaining increasing importance nowadays.

In this paper we study a few parameter distribution models of insurance costs. There are a number of characteristics that our collection of distributions should have. Through those it is another one describeing that some of them should have moderate tails, some should have heavy tails (see [4]).

The issue of the shape of the right-hand tail is a critical one for the actuary. If the possibility of large losses were remote, there might be little interest in the purchase of insurance. On the other hand, many loss processes produce large claims with enough

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regularity to inspire people and companies to purchase insurance and to create problems for the provider of insurance. For any distribution that extends probability to infinity, the issue is one of how quickly the density function approaches zero as the loss approaches infinity. The slower it happens, the more probability is pushed onto higher values and we then say the distribution has a heavy, thick or fat tails. We avoid the adjective „long" because in insurance a long tail usually means that it takes a long time from the issuance of the policy until all claims are settled.

Classical Exponential and Pareto distributions, used for insurance payments data, also the composite Exponential-Pareto has been studied in detail in [10]. Also, the comparative studies between Exponential, Pareto and Exponential-Pareto distributions has been made in [8] and [9]. Results regarding some characteristics of shifted composite Lognormal-Pareto and Weibull-Pareto as well the truncated ones, the probability density functions, has been presented in [7] . The composite Lognormal-Pareto and Weibull-Pareto models has been studied, in the literature, and used to model insurance payments data (see [2] , [1] and [5]).

1. Transformed Beta family

Many of the distributions presented in this section are specific cases of others. So, there is a family of distributions, named transformed Beta family, who are useful for modeling large claims. For example, a Burr distribution with gamma=1 and α and θ arbitrary is an Pareto distribution. Through this process, our distributions can be organized into one grouping as illustrated in Figure 1. The transformed Beta familiy indicates two special cases of a different nature. The paralogistic and inverse paralogistic distributions are created by setting the two nonscale parameters ao the burr and inverse burr distributions equal to each other rather than to a specific value.

1. Four-parameter transformed Beta distribution: $\alpha, \theta, \gamma, \tau$

$$f(x) = \frac{\Gamma(\alpha+\tau)}{\Gamma(\alpha)\Gamma(\tau)} \cdot \frac{\gamma (x/\theta)^{\gamma\tau}}{x\left[1+(x/\theta)^{\gamma}\right]^{\alpha+\tau}}$$

$$F(x) = \beta(\tau, \alpha; u), \quad \text{where} \quad u = \frac{(x/\theta)^{\gamma}}{1+(x/\theta)^{\gamma}}$$

2. Three-parameter distribution

2.1. Tree parameter generalized Pareto distribution: α, θ, τ

It is obtained from transformed Beta distribution, for $\gamma = 1$. Thus:

$$f(x) = \frac{\Gamma(\alpha+\tau)}{\Gamma(\alpha)\Gamma(\tau)} \cdot \frac{\theta^{\alpha} x^{\tau-1}}{(x+\theta)^{\alpha+\tau}}$$

$$F(x) = \beta(\tau, \alpha; u), \quad \text{where} \quad u = \frac{x}{x+\theta}$$

2.2. Tree-parameter Burr distribution: α, θ, γ

It is obtained from transformed Beta distribution, for $\tau = 1$. Thus:

$$f(x)=\frac{\alpha\gamma(x/\theta)^{\gamma}}{x\left[1+(x/\theta)^{\gamma}\right]^{\alpha+1}}$$

$$F(x)=1-u^{\alpha}\ , \quad \text{where} \quad u=\frac{1}{1+(x/\theta)^{\gamma}}$$

2.3. Tree-parameter inverese Burr distribution: θ,γ,τ

It is obtained from transformed Beta distribution, for $\alpha=1$. Thus:

$$f(x)=\frac{\tau\gamma(x/\theta)^{\gamma\tau}}{x\left[1+(x/\theta)^{\gamma}\right]^{\tau+1}}$$

$$F(x)=u^{\tau}\ , \quad \text{where} \quad u=\frac{(x/\theta)^{\gamma}}{1+(x/\theta)^{\gamma}}$$

3. *Two-parameter distributions*

3.1. Two-parameter Pareto distribution: α,θ

It is obtained from transformed Beta distribution, for $\gamma=\tau=1$, or from generalized Pareto, for $\tau=1$, or from Burr distribution, or $\gamma=1$. Thus:

$$f(x)=\frac{\alpha\theta^{\alpha}}{(x+\theta)^{\alpha+1}}$$

$$F(x)=1-\left(\frac{\theta}{x+\theta}\right)^{\alpha}$$

3.2. Two-parameter inverese Pareto distribution: τ,θ

It is obtained from transformed Beta distribution, for $\gamma=\alpha=1$, or from generalized Pareto distribution, for $\alpha=1$, or from inverse Burr distribution, for $\gamma=1$. Thus:

$$f(x)=\frac{\tau\theta x^{\tau-1}}{(x+\theta)^{\tau+1}}$$

$$F(x)=\left(\frac{x}{x+\theta}\right)^{\tau}$$

3.3. Two-parameter Loglogistic distribution: γ,θ

It is obtained from transformed Beta distribution, for $\tau=\alpha=1$, or from inverse Burr distribution $\tau=1$, or from Burr distribution for $\alpha=1$. Thus:

$$f(x)=\frac{\gamma(x/\theta)^{\gamma}}{x\left[1+(x/\theta)^{\gamma}\right]^{2}}$$

$$F(x)=u\ , \quad \text{where} \quad u=\frac{(x/\theta)^{\gamma}}{1+(x/\theta)^{\gamma}}$$

3.4 Two-parameter Paralogistic distribution: $\alpha\,,\theta$

It is obtained from transformed Beta distribution, for $\gamma=\alpha\,,\ \tau=1$ or from Burr distribution, for $\gamma=\alpha$. Thus:

$$f(x)=\frac{\alpha^{2}(x/\theta)^{\alpha}}{x\left[1+(x/\theta)^{\alpha}\right]^{\alpha+1}}$$

$$F(x)=1-u^{\alpha}\ , \quad \text{where} \quad u=\frac{1}{1+(x/\theta)^{\alpha}}$$

3.5. Two-parameter inverese Paralogistic distribution: $\tau\,,\theta$

It is obtained from transformed Beta distribution, for $\gamma=\tau\,,\ \alpha=1$ or from inverse Burr distribution, for $\gamma=\tau$. Thus:

$$f(x)=\frac{\tau^{2}(x/\theta)^{\tau^{2}}}{x\left[1+(x/\theta)^{\tau}\right]^{\tau+1}}$$

$$F(x)=u^{\tau}\ , \quad \text{where} \quad u=\frac{(x/\theta)^{\tau}}{1+(x/\theta)^{\tau}}$$

In conclusion, the distributions are represented as it follows:

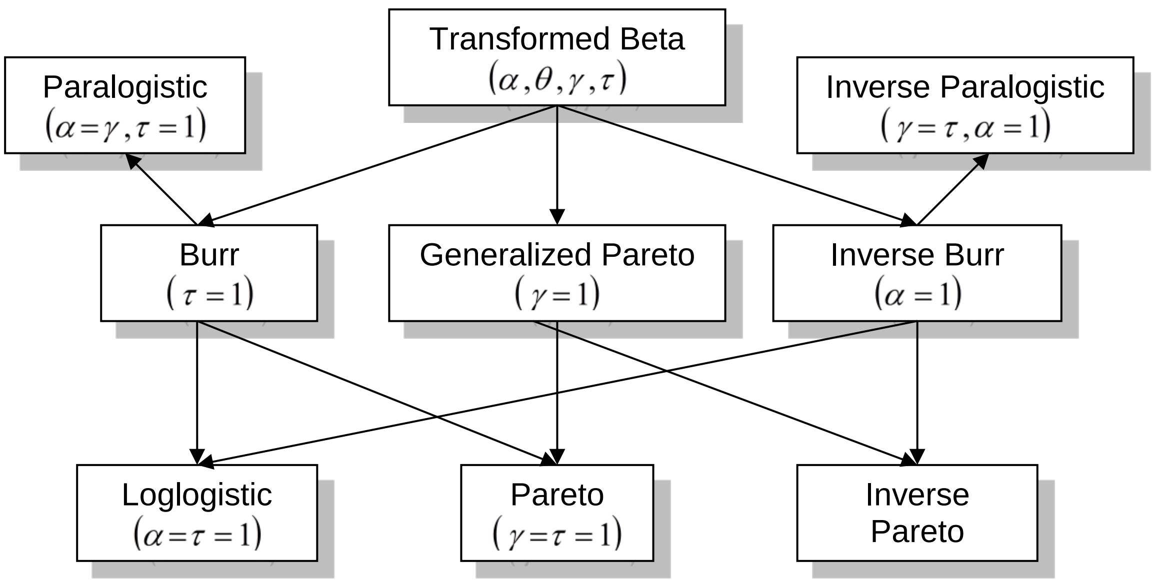

Fig. 1. Transformed Beta family

3. Tail behavior in the transformed Beta family

It is to be expected that with four parameters, the transformed Beta distribution can assume a variety of shapes which can be investigated by determining the relationships between the parameters and the tail behavior. For the right-hand tail we look at the existence of positive moments as well as the behavior of $1-F(x)$, for $x \to \infty$. We can provide the same analysis for the left-hand tail (that is, behavior for small claims).

3.1.Right tail behavior

For the transformed Beta distribution the k-th moment exist, for positive values of k, provided $k < \alpha\gamma$. Thus, for all members of the transformed Beta family, there are only a finite number of moments. However, it is clear that only the parameters α and γ are involved in determining the behavior at the right end. To further investigate this behavior, consider the ratio $[1-F(x)]/x^{-\alpha\gamma}$. When $x \to \infty$, both the numerator and denominator go to zero, thus we can apply L'Hôpital:

$$\lim_{x\to\infty} \frac{1-F(x)}{x^{-\alpha\gamma}} = \lim_{x\to\infty} \frac{-f(x)}{-\alpha\,\gamma\,x^{-\alpha\gamma-1}}$$

$$= \lim_{x\to\infty} \frac{\Gamma(\alpha+\tau)\,\gamma\,x^{\gamma\tau-1}\theta^{-\gamma\tau}}{\Gamma(\alpha)\Gamma(\tau)\left[1+\left(\dfrac{x}{\theta}\right)^{\gamma}\right]^{\alpha+\tau}\alpha\,\gamma\,x^{-\alpha\gamma-1}} = \lim_{x\to\infty} \frac{\Gamma(\alpha+\tau)}{\Gamma(\alpha)\Gamma(\tau)\theta^{\gamma\tau}\alpha}\left[\frac{\theta^{\gamma}x^{\gamma}}{\theta^{\gamma}+x^{\gamma}}\right]^{\alpha+\tau}.$$

For $x \to \infty$ the term in brackets goes to θ^{γ} and therefore:

$$\lim_{x\to\infty} \frac{1-F(x)}{x^{-\alpha\gamma}} = \frac{\Gamma(\alpha+\tau)\theta^{\alpha\gamma}}{\Gamma(\alpha)\Gamma(\tau)\alpha},$$

which is a constant and so the probability in the right-hand tail is approximately proportional to $x^{-\alpha\gamma}$. As a result, the behavior in the right tail is tied to the product $\alpha\gamma$. So, as the product $\alpha\gamma$ gets larger, the tail gets shorter (becomes zero faster).

Four illustrative density curves for the Burr distribution are presented in Figure 1.

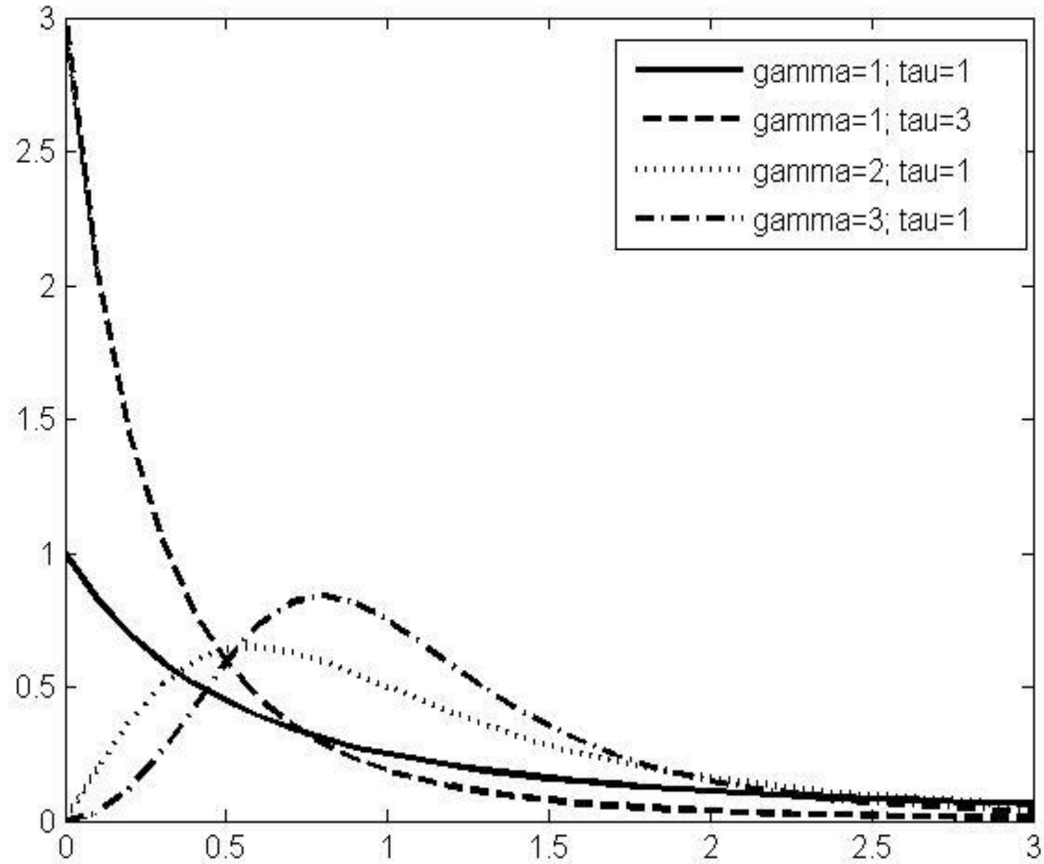

Fig.1 The Burr density curves plotted for $\theta = 1,\ \alpha = \tau,\ \gamma = \gamma$.

In Figure 1 we display few density curves for the Burr distribution. Notice that for this choice of parameters as α increases, the four densities approach zero faster.

The general case

Let ψ be a function, so that $\lim_{t\to 0}\psi(t) = 0$ and $\lim_{t\to 0}\frac{\psi(t)}{t} = \psi_0 \in \mathbf{R}$.

Then, $\lim_{x\to\infty}\frac{1-F(x)}{\psi\left(x^{-\alpha\gamma}\right)} = \lim_{x\to\infty}\frac{1-F(x)}{x^{-\alpha\gamma}}\cdot\frac{x^{-\alpha\gamma}}{\psi\left(x^{-\alpha\gamma}\right)} = \lim_{x\to\infty}\frac{\Gamma(\alpha+\tau)\theta^{\alpha\gamma}}{\Gamma(\alpha)\Gamma(\tau)\alpha}\cdot\frac{1}{\psi_0}$.

We can see that there are ψ functions that have the above-mentioned properties. For example, $\psi(t) = \psi_0 t$, $\psi(t) = \sin\psi_0 t$, $\psi(t) = tg\psi_0 t$.

So the probability in the right-hand tail is approximately proportional to $\psi\left(x^{-\alpha\gamma}\right)$, and the product $\alpha\gamma$ determines the right-hand tail behavior.

With regard to the limiting distributions, as α goes to infinity we obtained the transformed Gamma distribution. It has all the positive moments and so, all of its members have lighter tails than members of the transformed Beta family. As goes to infinity we obtain the inverse transformed Gamma distribution. The same limit ($\alpha\gamma$) applied to the existence of moments and so the right tail behavior is similar to that for the transformed Beta distribution. The inverse transformed Gamma distribution is obtained as τ goes to infinity and so all members of this family have heavy right tails whose behavior is governed by the product of two nonscale parameters.

3.2.Left tail behavior

The left tail behavior can be investigated by looking at the existence of negative moments. For the transformed Beta distributions more negative moments exist as the product $\gamma\tau$ gets larger, so it appears these parameters control the shape of the distribution for small loss values. With regard to probability in the left tail, $F(x) = P(X \le x)$, the requisite limit is:

$$\lim_{x\to 0}\frac{F(x)}{x^{\gamma\tau}} = \lim_{x\to 0}\frac{f(x)}{\gamma\,\tau\,x^{\gamma\tau-1}}$$

$$= \lim_{x\to 0}\frac{\Gamma(\alpha+\tau)\gamma\,x^{\gamma\tau-1}\theta^{-\gamma\tau}}{\Gamma(\alpha)\Gamma(\tau)\left[1+\left(\frac{x}{\theta}\right)^{\gamma}\right]^{\alpha+\tau}\gamma\,\tau\,x^{\gamma\tau-1}}$$

$$= \lim_{x\to 0}\frac{\Gamma(\alpha+\tau)}{\Gamma(\alpha)\Gamma(\tau)\tau\,\theta^{\gamma\tau}}\left[\frac{\theta^{\gamma}}{x^{\gamma}+\theta^{\gamma}}\right]^{\alpha+\tau}$$

$$= \frac{\Gamma(\alpha+\tau)}{\Gamma(\alpha)\Gamma(\tau)\tau\,\theta^{\gamma\tau}}.$$

The probability in the left-hand tail is approximately proportional to $x^{\gamma\tau}$. Again, the thickness of the tail is governed by the product $\gamma\tau$, and larger values indicate a thinner tail.

Conclusions

As a result, the behavior in the right tail of density in the transformed Beta family is tied to the product $\alpha\gamma$. So, as the product $\alpha\gamma$ gets larger, the tail gets shorter (becomes zero faster). In the left hand, the thickness of the tail is governed by the product $\gamma\tau$, and larger values indicate a thinner tail. It is interesting to note that the parameter γ affects the behavior of both tails (left and right). As γ increases, both tails get lighter and lighter. So, in some way, this parameter controls the spread around the middle, while the parameters α and τ control the right and left ends, respectively. As a scale parameter, θ plays no role in setting the shape of the distribution.

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